Regression characterizations in free and classical probability - a mysterious parallel

Jacek Wesołowski

Wydział Matematyki i Nauk Informacyjnych
Politechnika Warszawska

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Plan

1. Introduction
2. Lukacsian inspirations
3. A manual of regression characterizations

The regression characterization scheme

Let X and Y be independent/free random variables with distributions μ and ν, respectively. Let ψ be such a function that for $(U, V) = \psi(X, Y)$ and for integer s_i there exists $c_i \in \mathbb{R}$ such that

$$\mathbb{E}(U^{s_i}|V) = c_i \quad \text{for} \quad i = 1, 2. \tag{1}$$

A characterization idiom:

Assume that X and Y are independent/free and (1) holds.

Are distributions of X and Y necessarily μ and ν, respectively?

Independence/freeness characterization scheme

Let X and Y be independent/free random variables with distributions μ and ν, respectively. Let ψ be such a function that U and V, defined by $(U, V) = \psi(X, Y)$, are independent/free.

A characterization idiom:

Assume that X and Y are independent/free and U and V are also independent/free.

Are distributions of X and Y necessarily μ and ν, respectively?

Bernstein/Nica independence/freeness characterization

Let X and Y be independent/free random variables with common distributions Gaussian/Wigner distribution. Then $U = X - Y$ and $V = X + Y$ are independent/free.

Characterization: If X and Y are independent/free and $U = X - Y$ and $V = X + Y$ are independent/free then X and Y have the common Gaussian/Wugner Wigner distribution.

Regression version of Bernstein/Nica characterization

Let X and Y be independent/free random variables with common distributions Gaussian (Wigner) distribution.
Since $X - Y$ and $X + Y$ are independent/free

$$\mathbb{E}(X|X+Y) = \frac{X+Y}{2} \tag{2}$$

$$\mathbb{E}(X^2|X+Y) = \frac{(X+Y)^2}{4} + C \tag{3}$$

Characterization: If X and Y are independent/free and (2) and (3) hold then X and Y have the common Gaussian/Wigner distribution.

Laha-Lukacs/Bożejko-Bryc regression characterizations of free Meixner laws

Let X and Y be independent/free random variables with zero means. Assume that

$$\mathbb{E}(X|X+Y) = \alpha(X+Y) \tag{4}$$

$$\mathbb{V}\mathrm{ar}(X|X+Y) = C(1 + a(X+Y) + b(X+Y)^2). \tag{5}$$

Then distributions of X and Y are of free Meixner type:
(a) Gaussian/Wigner if $a = b = 0$;
(b) Poisson/Marchenko-Pastur if $a \neq 0$ and $b = 0$;
(c) free gamma if $a^2 = 4b > 0$;
(d) free negative binomial if $b > 0$ and $a^2 > 4b$;
(e) free binomial if $-\min\{\alpha, 1-\alpha\} \leq b < 0$;
(f) hyperbolic secant (free pure Meiner) if $0 < a^2 < 4b$.

Lukacs type characterization and property in free probability

A corollary of **(b)** in free probability:

Prop. 3.5 [Bożejko, Bryc (2006)] Let X and Y be free and $X + Y > 0$. If

$$V = X + Y \quad \text{and} \quad U = (X + Y)^{-1/2} X (X + Y)^{-1/2}$$

are free then X and Y have free-Poisson laws.

Then V has also a Poisson law and U has a free binomial distribution.

Is the converse true?

Lukacs type characterization in classical probability

A corollary of **(c)** in classical probability:

Th. [Lukacs (1955)]. Let X and Y be independent and positive random variabes. Then $V = X + Y$ and $U = X/(X + Y)$ are independent iff X and Y have gamma laws with the same scale, i.e. the densities are of the form

$$f(x) \propto x^{p-1} e^{-ax} I_{(0,\infty)}(x).$$

A dual version (trivial!): Let U and V be independent random variables, $V > 0$, $0 < U < 1$. Then $X = UV$ and $Y = (1 - U)V$ are independent iff V has a gamma law and U has a beta law, where the beta density is of the form

$$g(x) \propto x^{p-1}(1 - x)^{q-1} I_{(0,1)}(x).$$

Dual Lukacs type regressions:

Let U and V be independent/free , U supported in $[0,1]$, V compactly supported in $(0,\infty)$. Let

$$X = UV \qquad (X = V^{1/2}UV^{1/2})$$

and

$$Y = (1-U)V \qquad (Y = V^{1/2}(1-U)V^{1/2}).$$

If for one of pairs $(s_1, s_2) \in \{(1,2), (1,-1), (-1,-2)\}$

$$\mathbb{E}(Y^{s_i}|X) = c_i, \qquad i = 1,2,$$

then V has a gamma)free Poisson) distribution and U has a beta/(free binomial) distribution.

Bobecka & JW (2002): $(1,2)$, $(1,-1)$, $(-1,-2)$
Szpojankowski & JW (2014): $(1,2)$,
Szpojankowski (2014): $(1,-1)$, $(-1,-2)$.

Dual Lukacs independence in free probability, Szpojankowski & JW (2014)

Theorem

Let U and V be free, U supported in $(0,1)$ and V supported compactly in $(0,\infty)$. Define

$$X = V^{1/2}UV^{1/2} \quad \text{and} \quad Y = V^{1/2}(1-U)V^{1/2}.$$

1. *If X and Y are free then V and U have (special) free Poisson and free binomial distributions, respectively.*
2. *If V and U have (special) free Poisson and free binomial distributions, respectively, then X and Y are free (with suitable free Poisson distributions).*

The first statement follows from the regression characterization.

Second statement:

- By asymptotic freeness (Captaine and Casalis, 2004) there exist (suitable) $n \times n$ independent beta, $\mathbf{U}_n$, and Wishart, $\mathbf{V}_n$, matrices, $n \geq 1$, such that for any polynomial Q

$$\lim_{n\to\infty} \mathbb{E}_n(Q(\mathbf{U}_n, \mathbf{V}_n)) = \mathbb{E}\, Q(U, V),$$

  where $\mathbb{E}_n(\cdot) = n^{-1}\mathbb{E}\,\mathrm{tr}(\cdot)$.

- For any $n \geq 1$ random matrices

$$\mathbf{X}_n = \mathbf{V}_n^{1/2}\mathbf{U}\mathbf{V}_n^{1/2} \quad and \quad \mathbf{Y}_n = \mathbf{V}_n - \mathbf{V}_n^{1/2}\mathbf{U}\mathbf{V}_n^{1/2}$$

  are independent Wishart (e.g. Olkin and Rubin, 1964), Casalis, Letac, 2004)

Second statement, cont.:

- Due to asymptotic freenes of $(\mathbf{X}_n, \mathbf{Y}_n)$ for any polynomial P

$$\lim_{n\to\infty} \mathbb{E}_n P(\mathbf{X}_n, \mathbf{Y}_n) = \mathbb{E} P(X', Y'),$$

  where X' and Y' are free with (suitable) free Poisson distributions.
- Fix any ploynomial P. By traciality there exists a polynomial Q such that

$$\mathbb{E}_n P(\mathbf{X}_n, \mathbf{Y}_n) = \mathbb{E}_n Q(\mathbf{U}_n, \mathbf{V}_n) \to \mathbb{E} Q(U, V) = \mathbb{E} P(X, Y)$$

- Consequently, for any polynomial P

$$\mathbb{E} P(X', Y') = \mathbb{E} P(X', Y').$$

Recall:

Prop. 3.5 [Bożejko, Bryc (2006)] Let X and Y be free and $X + Y > 0$. If

$$V = X + Y \quad \text{and} \quad U = (X + Y)^{-1/2} X (X + Y)^{-1/2}$$

are free then X and Y have free-Poisson laws.

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Is the converse true?

Theorem

Let X and Y be free with (suitable) free Poisson distributions. Then V and U are free.

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Yes, it is true!

The first, combinatorial, proof based on direct calculation of joint free cumulants of U and V was given in Szpojankowski (2015). The highlight of the argument was the explicit formula for joint free cumulants of X and X^{-1} when X is a free Poisson variable with the rate λ and the jump size 1:

$$\mathcal{R}_{i_1+\ldots+i_m+m}(X^{-1},\underbrace{X,\ldots,X}_{i_1},X^{-1},\underbrace{X,\ldots,X}_{i_2},X^{-1},\ldots,X^{-1},\underbrace{X,\ldots,X}_{i_m})$$

$$=\begin{cases} 0, & \exists\, k:\ i_k>1,\\ (-1)^{i_1+\ldots+i_m}\mathcal{R}_m(X^{-1}), & \forall\, k,\ i_k\le 1\end{cases}$$

and

$$\mathcal{R}_m(X^{-1})=\frac{C_{m-1}}{(\lambda-1)^{2m-1}},$$

where C_n is the nth Catalan number.

A sketch of a new proof based on Theorem 1:

- Let $\tilde{U}$, $\tilde{V}$ be free with suitable free binomial and free Poisson distributions. Define

$$\tilde{X} = \tilde{V}^{1/2} U \tilde{V}^{1/2} \quad \text{and} \quad \tilde{Y} = \tilde{V} - \tilde{V}^{1/2} U \tilde{V}^{1/2}.$$

- By the second part of Theorem 1 $\tilde{X}$ and $\tilde{Y}$ are free. Moreover, distributions of X and $\tilde{X}$ are the same, and distributions of Y and $\tilde{Y}$ are identical.
- For any polynomial P there exists function g such that $P(U, V) = g(X, Y)$. By defintion of $\tilde{X}$ and $\tilde{Y}$ we have also $P(\tilde{U}, \tilde{V}) = g(\tilde{X}, \tilde{Y})$.
- For X and Y free $\mathbb{E}\, g(X, Y)$ depends only on distributions of X and Y. Therefore, $\mathbb{E}\, g(\tilde{X}, \tilde{Y}) = \mathbb{E}\, g(X, Y)$.
- Consequently, $\mathbb{E}\, P(\tilde{U}, \tilde{V}) = \mathbb{E}\, P(U, V)$.

Direct moment approach

Eg. consider

$$\mathbb{E}(V - V^{1/2}UV^{1/2} | V^{1/2}UV^{1/2}) = a$$

and

$$\mathbb{E}((V - V^{1/2}UV^{1/2})^2 | V^{1/2}UV^{1/2}) = b.$$

- Multiply both sides by $(V^{1/2}UV^{1/2})^n$, $n \geq 1$, and take $\mathbb{E}$ of both sides. Then, by traciality,

$$\beta_n - \alpha_{n+1} = a\alpha_n \quad \text{and} \quad \gamma_n - 2\beta_{n+1} + \alpha_{n+2} = b\alpha_n,$$

  where $\alpha_n = \mathbb{E}\,(VU)^n$, $\beta_n = \mathbb{E}\,V(VU)^n$, $\gamma_n = \mathbb{E}\,V^2(VU)^n$.
- For generating functions A, B and C of (α_n), (β_n) and (γ_n)

$$B(z) - \frac{A(z)-1}{z} = aA(z) \tag{6}$$

  and

$$C(z) - \frac{2z(B(z)-\beta_0)+A(z)-\alpha_1 z-1}{z^2} = bA(z). \tag{7}$$

Direct moment approach, cont.

- Let D be the generating function of $(\delta_n = \mathbb{E}\, U(VU)^n)$, $T(z) = zD(z)$, r the r-transform of V and $R = r \circ T$. Then

$$A = 1 + T\,R, \quad B = TR(1+R), \quad C = R(R - \beta_0) + \frac{R-\beta_0}{T}.$$

- Plug in such A, B and C into (6) and (7). After some algebra, with $h = TR = M_{UV}$, one gets

$$h(1 - \alpha T) = \lambda \alpha T, \tag{8}$$

and

$$zh(z) + z = \frac{\lambda \alpha T(z)}{a(\alpha T(z) - 1) + \lambda\alpha}. \tag{9}$$

- (8) yields $r(z) = \frac{\lambda\alpha}{1-\alpha z}$, i.e. V has the free Poisson law.
- Then both (8) and (9) allow to identify $\psi_{UV} := h^{-1}$ and thus the S-transform

$$S_{UV}(z) = (\lambda\alpha - a + \alpha z)^{-1}.$$

- Since $S_{UV} = S_U\, S_V$ and S_{UV}, S_V are known, S_U is identified as the S-transform of free a beta law.

Subordination approach

Orignally proposed in Ejsmont, Franz & Szpojankowski (2017). Here we apply it to dual regressions with $(s_1, s_2) = (1, -1)$:

$$\mathbb{E}(V - V^{1/2}UV^{1/2}|V^{1/2}UV^{1/2}) = a$$

and

$$\mathbb{E}((V - V^{1/2}UV^{1/2})^{-1}|V^{1/2}UV^{1/2}) = c.$$

By subordination for $\psi_W(z) := zW(1 - zW)^{-1}$ there exist functions ω_1 and ω_2 such that

$$\mathbb{E}(\psi_{V^{1/2}UV^{1/2}}(z)|V) = \psi_V(\omega_1(z)) \quad \text{and} \quad \mathbb{E}(\psi_{U^{1/2}VU^{1/2}}(z)|U) = \psi_U(\omega_2(z)).$$

Since $M_W(z) = \mathbb{E}\,\psi_W(z)$ we have

$$M_{UV}(z) = M_V(\omega_1(z)) = M_U(\omega_2(z)).$$

Subordination technique

- Multiply both sides of regressions by $\psi_{V^{1/2}UV^{1/2}}(z)$ and apply $\mathbb{E}$ together with traciality to get

$$\begin{cases} K := \mathbb{E}\,(1-U)\mathbf{V^{1/2}}\psi_{\mathbf{V^{1/2}UV^{1/2}}}(\mathbf{z})\mathbf{V^{1/2}} = aM_{UV(z)}, \\ L := \mathbb{E}\,(1-U)^{-1}\mathbf{V^{-1/2}}\psi_{\mathbf{V^{1/2}UV^{1/2}}}(\mathbf{z})\mathbf{V^{-1/2}} = bM_{UV(z)} \end{cases}$$

- Note the algebraic identity

$$\mathbf{W}^{-1/2}\psi_{\mathbf{W^{1/2}TW^{1/2}}}(\mathbf{z})\mathbf{W}^{-1/2} = \mathbf{zT}^{1/2}\psi_{\mathbf{T^{1/2}WT^{1/2}}}(\mathbf{z})\mathbf{T}^{1/2} + \mathbf{zT}. \quad (10)$$

- Plug (10) with $(W, T) := (U, V)$ into K and with $(W, T) := (V, U)$ into L:

$$\begin{cases} K = \mathbb{E}\,(1-U)U^{-1/2}\Psi_{U^{1/2}VU^{1/2}}(z)U^{-1/2} - z\mathbb{E}\,(1-U)V, \\ L = z\mathbb{E}\,(1-U)^{-1}U^{1/2}\Psi_{U^{1/2}VU^{1/2}}(z)U^{1/2} + z\mathbb{E}\,U(1-U)^{-1}. \end{cases}$$

Subordination technique, cont.

- Conditioning with respect to U and using subordination we finally get ($\alpha = \mathbb{E}\,\psi_U(1)$)

$$\begin{cases} \omega_2(z) + (\omega_2(z) - 1)M_U(\omega_2(z)) = az(M_U(\omega_2(z)) + 1), \\ z(M_U(\omega_2(z)) - \alpha) = b(\omega_2(z) - 1)M_U(\omega_2(z)). \end{cases} \tag{11}$$

- For $H_U = M_U^{-1}$ and $H_{UV} = M_{UV}^{-1}$ we get

$$\begin{cases} a(1+s)H_{UV}(s) = (1+s)H_U(s) - s, \\ (s-\alpha)H_{UV}(s) = bs(H_U(s) - 1). \end{cases}$$

- Solving this system gives the S-transforms

$$\begin{cases} S_U(s) = \frac{1+s}{s}H_U(s) = 1 + \frac{ab}{\alpha+ab+(ab-1)s}, \\ S_{UV}(s) = \frac{1+s}{s}H_{UV}(s) = \frac{b}{\alpha+ab+(ab-1)s}. \end{cases}$$

- Finally, S_V follows from $S_{UV} = S_U S_V$.

A difficulty in regressins with $(s_1, s_2) = (-1, -2)$

Consider

$$\begin{cases} \mathbb{E}((V^{1/2}(1-U)V^{1/2})^{-1} | V^{1/2}UV^{1/2}) = b, \\ \mathbb{E}((V^{1/2}(1-U)V^{1/2})^{-2} | V^{1/2}UV^{1/2}) = c. \end{cases} \tag{12}$$

- From the second condition in (12)

$$N := \mathbb{E}\,(1-U)^{-1}V^{-1}(1-U)^{-1}\mathbf{V}^{-1/2}\psi_{\mathbf{V^{1/2}UV^{1/2}}}\mathbf{V}^{-1/2} = cM_{UV}. \tag{13}$$

- Identity (10) with $(W, T) = (V, U)$ gives

$$\begin{aligned} N =& z\mathbb{E}\,V^{-1}(1-U)^{-1}U^{1/2}\psi_{U^{1/2}VU^{1/2}}(z)U^{1/2}(1-U)^{-1} \\ &+ z\mathbb{E}\,V^{-1}(1-U)^{-2}U \\ =& z\mathbb{E}\,V^{-1}\mathbb{E}((\mathbf{1}-\mathbf{U})^{-\mathbf{1}}\mathbf{U}^{\mathbf{1/2}}\psi_{\mathbf{U^{1/2}VU^{1/2}}}(\mathbf{z})\mathbf{U}^{\mathbf{1/2}}(\mathbf{1}-\mathbf{U})^{-\mathbf{1}}|\mathbf{V}) \\ &+ z\mathbb{E}\,V^{-1}(1-U)^{-2}U. \end{aligned}$$

Boolean cumulants to rescue

- **Boolean cumulants** help in calculating this conditional exepectation:

$$\mathbb{E}((1-U)^{-1}U^{1/2}\psi_{U^{1/2}VU^{1/2}}(z)U^{1/2}(1-U)^{-1}|V)$$
$$= B_2(z) + zB_1^2(z)(1+\psi_V(\omega_1(z))),$$

where

$$B_1(z) = \tfrac{\eta_U(\omega_2(z))-\eta_U(1)}{\omega_2(z)-1}\mathbb{E}\,(1-U)^{-1},$$

$$B_2(z) = \frac{\omega_2(z)\left[\eta_U(\omega_2(z))-\eta_U(1)-(\omega_2(z)-1)\eta_U'(1)\right]}{(\omega_2(z)-1)^2}\,\mathbb{E}^2(1-U)^{-1},$$

and $\eta_U = \frac{M_U}{1+M_U}$ is the generating function of the sequence of Boolean cumulants of U.

- Thus (13) assumes the form

$$\left(\tfrac{z}{b(\omega_2-1)}\right)^2 \tfrac{M_U(\omega_2)-\alpha}{M_U(\omega_2)+1}\,\left[b\omega_2 - z(M_U(\omega_2)-\alpha)\right]b^2 \qquad (14)$$
$$= cz\left(\alpha\tfrac{z}{b(\omega_2-1)} + M_U(\omega_2)\right).$$

Final touch

- The first regression condition, see the second equation of (11) in the previous regression problem, leads to

$$\frac{z}{b(\omega_2-1)} = \frac{M_U(\omega_2)}{M_U(\omega_2)-\alpha}. \tag{15}$$

  Plugging (15) into (14) gives

$$\omega_2 + (\omega_2 - 1)M_U(\omega_2) = \frac{c}{b^3} z(M_U(\omega_2) + 1). \tag{16}$$

- Thus the system of equations (16) and (15) is the same as (11) with $a := \frac{c}{b^3}$.

Mysteries

- **THE SAME regression characterizations in classical and free probability**
- **In Lukacs regressions**
  free binomial $\equiv$ beta
  free Poisson $\equiv$ gamma

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